A Dynamical Approach to the Evolution of Complex Networks

Thomas E. Gorochowski, Mario di Bernardo and Claire S. Grierson

***Abstract*— In this work we take a dynamical approach to the evolution of complex networks using simulated output of the full system dynamic to direct evolution of the underlying network structure. Extending previous work, we study the problem of enhanced synchronisation and the generality of Type 2 features which have been shown to emerge in regimes where full synchronisation is unstable. Networks are evolved using a new computational tool called NetEvo which aims to minimise a dynamical order parameter performance measure. This process is performed for networks with several alternative node dynamics, showing in all cases that qualitatively similar Type 2 topologies emerge. Analysis of these structures highlights variation in many of the network statistics and motif frequencies, but helps to classify some key characteristics exhibited by all Type 2 networks, regardless of node dynamic.**

I. INTRODUCTION

Over recent years many types of complex system have been described using dynamical networks as a basis (e.g. see [1]); nodes represent components having an internal dynamic and edges dictate how these components interact with one another. Such a description provides a common framework in which to study many types of complex phenomena, with synchronisation having seen much attention. Synchronisation forms an important part of many natural and man-made processes, ranging from circadian rhythms in biology [2], to oscillating voltages across power grids [3].

When studying synchronisation it is common to consider a network of N coupled m-dimensional systems of the form,

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) - \sigma \sum_{j=1}^{N} \mathcal{L}_{ij} \mathbf{H} \mathbf{x}_j, \quad i = 1, 2, \ldots, N, \qquad (1)$$

where $\mathbf{x}_i \in \mathbb{R}^m$ is the state of the ith node, $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^m$ defines the internal node dynamics, $\sigma \in \mathbb{R}$ is the global coupling strength between connected nodes, $\mathcal{L} = (\mathcal{L}_{ij})$ is the network Laplacian, and $\mathbf{H} \in \mathbb{R}^{m \times m}$ is a matrix specifying the inner coupling between states of two interacting nodes. We say a system is *fully* or *asymptotically synchronised* if $\lim_{t\to\infty} \|\mathbf{x}_i - \mathbf{x}_j\| = 0$.

It was shown by Pecora and Carroll in [4] that the *synchronisability* of a network—the range of coupling strengths for which full synchronisation is achieved—is related to the eigenvalues of the network Laplacian. They illustrated how the *master stability function* (MSF) can be formulated for dynamical networks, and how minimisation of the *eigenratio* (λ_N/λ_2) ensures stability of the synchronisation manifold, $\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_N$, for the largest range of coupling strengths [5]. More recently, this work has also formed the basis of attempts to improve the synchronisability of networks, by finding methods to rewire the structure such that the eigenratio is minimised. These have included network optimisation techniques [6] and the utilisation of graph theoretic knowledge [7], [8] to help make better decisions when altering the underlying topology. Optimised networks produced using these methods were highly homogeneous, displaying short path lengths and narrow degree and betweenness distributions. These networks were referred to as "entangled" in [6], with an example shown in Fig. 1(a).

The eigenratio is classified as a purely *topological measure*, requiring no consideration of system-level dynamics when being calculated. This permits widespread use across many types of system. Such generality is a major benefit of topological measures, however, there are limitations related to possible constraints on the system of interest. For example, it may be that the coupling strength of the system is fixed at a value that does not allow for a stable synchronisation manifold. In this case, minimisation of the eigenratio may provide no benefit at all. Furthermore, a more specific form to the synchronisation may be required, for example only occurring at a specified coupling strength. Under these sorts of scenario, where existing topological measures are unlikely to exist, a more flexible approach is required.

In [9] we attempted to use a dynamical approach to improving synchronisation. A computational tool called NetEvo[1] [10] was developed to evolve the network structure of a system through the rewiring of edges. The evolutionary process was embodied in an optimisation method, using a *dynamical measure* (order parameter) based on simulated output of the full system dynamic. Networks were evolved using Rössler node dynamics and diffusive coupling. Furthermore, a variety of fixed coupling strengths were used to see how this would affect the final topologies. It was shown that for coupling strengths permitting stable synchronisation, structures converged towards "entangled" features generating what was termed Type 1 networks [see Fig. 1(a)].

As the fixed coupling strength was increased we reached a point at which the synchronisation manifold became unstable, reflected in a change to the evolved topologies. A transition was observed to networks displaying hub-like

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[1]For further information about the framework, see the project website at http://www.netevo.org

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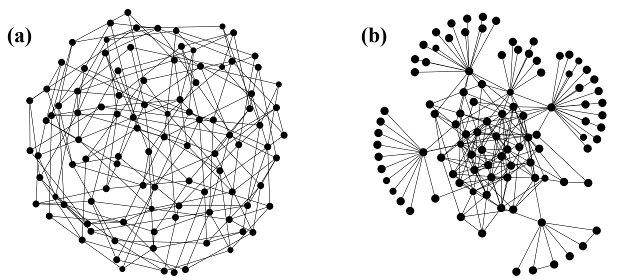


Fig. 1. The two types of evolved topology. (a) Type 1 “entangled” and (b) Type 2.

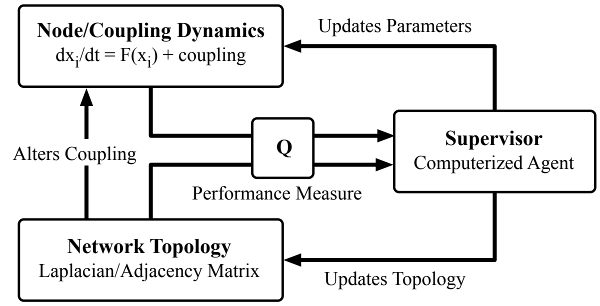


Fig. 2. Flowchart of a supervised network taken from [9].

features, surrounding a highly interconnected central region. These were classified as Type 2 networks and the transition as a *topological bifurcation*. An example of a Type 2 network can be found in Fig. 1(b). Furthermore, Type 2 structures were shown to have an increased frequency of 3 node closed loop feedback motifs and it was conjectured that these may allow for localised stability during the evolutionary process.

This previous work focused on assessing the viability of a dynamical approach when evolving improved topologies for synchronisation. In doing so, it did not fully investigate the generality of the results to alternative node dynamics, concentrating on a single form throughout. In this paper we build on this work, attempting to understand the sensitivity of the emergence of Type 2 topologies to alternative forms of node dynamic.

II. THE NETEVO FRAMEWORK

NetEvo [10] is a computational framework developed to allow for numerical experimentation and analysis of evolving dynamical networks. To carry out the evolutionary process, NetEvo introduces the idea of a *supervisor* which attempts to make changes to the underlying topology and dynamical parameters, such that a user specified performance measure Q is minimised. The concept of a supervised network is illustrated in Fig. 2.

In many ways this process can be viewed as an optimisation procedure, searching for dynamical networks with improved performance. For this reason, NetEvo currently implements a *simulated annealing* metaheuristic to search for near-optimal configurations. Furthermore, the framework has been built in an extendible manner to allow for users to specify their own node and edge dynamics, evolutionary processes and performance measures. At present, the form of dynamics is limited to continuous ordinary differential equations, however, we hope to extend this to discrete and stochastic processes in the near future.

III. SENSITIVITY TO NODE DYNAMICS

To assess the sensitivity of the Type 2 topologies to node dynamics, we used NetEvo to evolve networks for a variety of node types. All networks consisted of 100 nodes with an average degree of 4 and an initial random topology. The number of nodes and edges remaind fixed throughout evolution, with only rewiring of edges permitted.

In the following sections we describe the types of node dynamics and coupling that were chosen, and the method of network evolution.

A. System Dynamics

Three types of network were considered, each with an alternative form of identical node dynamic, to assess the impact this had on resultant topological features. We defined each node to have an internal state $\mathbf{x}_i = (p_i, q_i, r_i)$, with the dynamics of either a standard Rössler, Lorenz or Chua oscillator described by the equations,

$$\mathbf{F}_{\text{Rössler}}(\mathbf{x}_i) = \begin{cases} -q_i - r_i \\ p_i + 0.165q_i \\ 0.2 + (p_i - 10)r_i, \end{cases} \quad (2)$$

$$\mathbf{F}_{\text{Lorenz}}(\mathbf{x}_i) = \begin{cases} 10(q_i - p_i) \\ p_i(28 - r_i) - q_i \\ p_i q_i - 2r_i, \end{cases} \quad (3)$$

$$\mathbf{F}_{\text{Chua}}(\mathbf{x}_i) = \begin{cases} 10[q_i - p_i + g(p_i)] \\ p_i - q_i + r_i \\ -14.87q_i, \end{cases} \quad (4)$$

where,

$$g(x) = \begin{cases} 0.68x + 1.27 - 0.68, & x > 1 \\ 1.27x, & |x| < 1 \\ 0.68x - 1.27 + 0.68, & x < -1, \end{cases}$$

Diffusive coupling was used between nodes with internal states of connected nodes i and j varying for each type of network; Rössler based networks used $p_i \rightarrow p_j$ and $r_i \rightarrow r_j$, Lorenz based networks used $q_i \rightarrow p_j$, and Chua based networks used $r_i \rightarrow r_j$. These coupling schemes were selected to ensure a master stability function of the form Γ_2 as reported in the literature [11], leading to a bounded region of stability (α_1, α_2) for the synchronisation manifold. Master stability functions for each type of system are shown in Fig. 3. Large fixed coupling strengths $\hat{\sigma}$ were selected during evolution to place the systems into regimes where full synchronisation was not possible, i.e. where $\hat{\sigma}\lambda_N > \alpha_2$. This allowed us to specifically investigate the generality of Type 2 topologies which had previously been seen to emerge within this regime. The chosen fixed coupling strengths were $\hat{\sigma} = 0.9$ for Rössler based networks, $\hat{\sigma} = 4.5$ for Lorenz based networks, and $\hat{\sigma} = 0.85$ for Chua based networks.

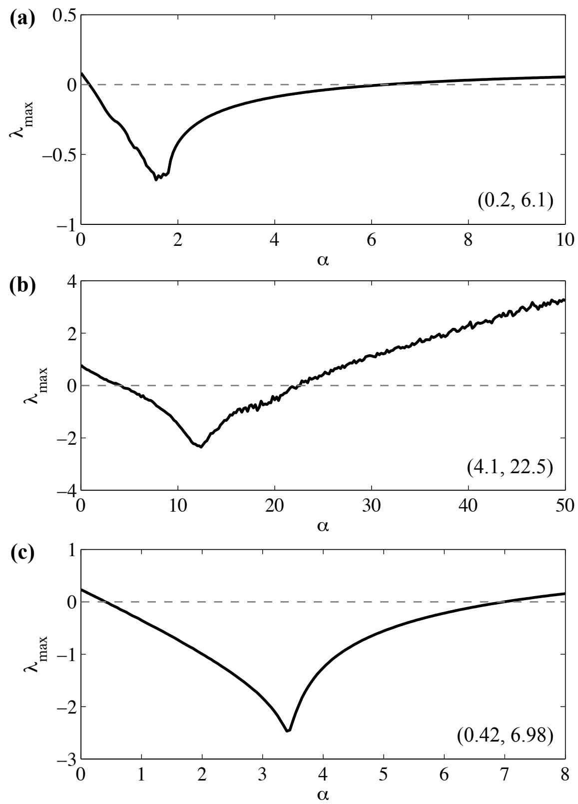


Fig. 3. Master stability functions for our chosen (a) Rössler, (b) Lorenz, and (c) Chua systems. An approximate stable range (α_1, α_2) is displayed at the bottom right of each graph.

To simulate system dynamics during the evolutionary process, NetEvo was configured to use an Embedded Runge-Kutta-Fehlberg (4, 5) numerical method with an adaptive step size and absolute and relative errors of 10^{-5}. To minimise the effect of finite simulation times an average of 10 separate runs was taken, with each simulation lasting for 150 time units.

B. Network Evolution

Networks were evolved using NetEvo's simulated annealing supervisor configured in a similar way to [9]. Evolution started with 100 random networks being generated and an initial temperature for the optimisation procedure being calculated using $4q_{\text{max}}$, where q_{max} was the maximum difference in the performance measure Q for each of the initial networks. Temperature remained fixed for every trail at a given step and was reduced by 10% once a step had been completed. Every step consisted of a maximum 5000 trails, or 500 consecutively accepting trails. For each trail a number of edges, selected from an exponentially distributed random variable with a mean value of 1, were rewired and the performance measure recalculated. If this lead to an improvement then the new configuration was selected. If not, then the new configuration was selected with a probability $e^{-dQ/T}$, where dQ was the change in performance measure and T was the current temperature. Evolution halted after 5 temperature reductions without change, if 300000 trials had been performed, or if a temperature of 10^{-7} was reached.

To direct the evolutionary process we adapted the dynamical *order parameter* performance measure of Yook and Meyer-Ortmanns [12] taking,

$$Q_{\text{OP}} = 1 - \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j=1}^{N} \Theta \left[\delta - d_{ij}(t_\infty) \right], \quad (5)$$

where N is the number of nodes, Θ is the Heaviside function, δ is a threshold to accommodate numerical errors during simulation, d_{ij} is the standard Euclidean norm distance between trajectories for nodes i and j, and t_∞ is a time at which trajectories will have converged to within δ if at all possible.

IV. ANALYSIS OF THE EVOLVED TOPOLOGIES

Networks were evolved using Rössler, Lorenz and Chua node dynamics. The resultant topologies were analysed by calculating standard statistical measures and motif distributions, searching of similarities and differences between their features. To provide a reference for comparison, previous results for Type 1 topologies were included in the analysis.

A. Topological Structure

To better understand the evolved topologies, visualisations and average statistical measures were calculated (see Fig. 4 and Table I). From these results, two main types of evolved structure were immediately evident. As reported in [9], topologies (a) and (b) showed that topological and dynamical performance measures lead to a convergence of similar Type 1 "entangled" structures, when the fixed coupling strength $\hat{\sigma}$ used during evolution allowed for a stable synchronisation manifold, i.e. when $\hat{\sigma}\lambda_2 > \alpha_1$ and $\hat{\sigma}\lambda_N > \alpha_2$. Visualisations show a highly homogeneous and interwoven topology, with no discernible features separating the structure despite the different node dynamics. Furthermore, statistics reveal that, in all cases, both dynamical and topological approaches lead to reduced network diameter and eigenratio.

Figures 4(c), (d) and (e) show our newly evolved topologies, generated using a fixed coupling strength that results in an unstable synchronisation manifold. These display very different structures, characterised by standard Type 2 features – hubs connected to lower degree nodes, surrounding a highly interconnected central region. Although all Type 2 networks are qualitatively similar, specific differences can be seen for each type of node dynamic.

Type 2 Rössler based networks contain the largest hubs, consisting mainly of one and two degree nodes. The size of these hubs leads to a surplus of edges, as the total number remains fixed during evolution with only rewiring allowed. This provides necessary resources for an increased density at the interconnected central region, helping reduce network diameter to 5.60 and increase the average clustering coefficient to 0.0185.

Lorenz networks display the weakest Type 2 form, with small hubs containing single degree nodes. This reduction in hub size causes a less densely packed central region and

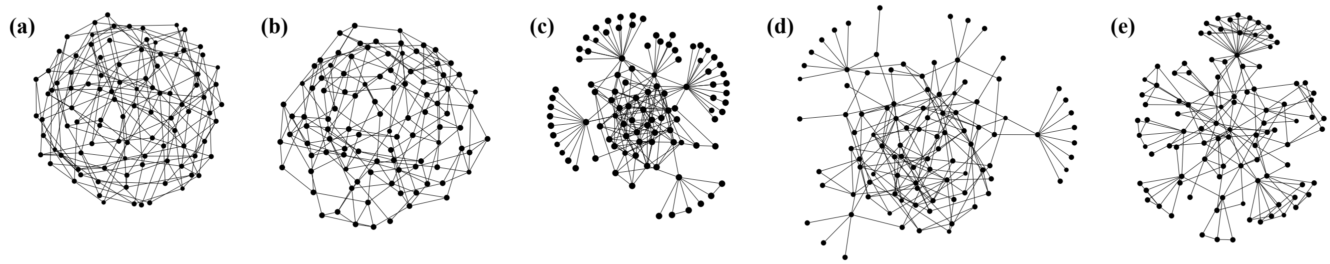


Fig. 4. Evolved topologies for both toplogical (Eigenratio) and dynamical (Order Parameter) performance measures with differing node dynamics. (a) Eigenratio, (b) Rössler – Order Parameter $\hat{\sigma} = 0.6$, (c) Rössler – Order Parameter $\hat{\sigma} = 0.9$, (d) Lorenz – Order Parameter $\hat{\sigma} = 4.5$, and (e) Chua – Order Parameter $\hat{\sigma} = 0.85$. It is clear to see the two qualitatively different topologies with (a) and (b) displaying Type 1 and (c) to (e) Type 2 features.

TABLE I
STATISTICS FOR THE EVOLVED TOPOLOGIES

			Diameter		**Clustering**		**Eigenratio**	
Dynamics	**Performance Measure**	**Topology**	Average	Std Dev	Average	Std Dev	Average	Std Dev
–	Eigenratio	Type 1	5.25	0.4443	0.0029	0.0036	7.57	0.11
Rössler	Order Parameter $\hat{\sigma}$ = 0.6	Type 1	6.20	0.4104	0.0917	0.0175	12.53	0.58
Rössler	Order Parameter $\hat{\sigma}$ = 0.9	Type 2	5.60	0.5477	0.0798	0.0185	154.73	53.02
Lorenz	Order Parameter $\hat{\sigma}$ = 4.5	Type 2	7.30	0.8233	0.0466	0.0131	55.63	31.35
Chua	Order Parameter $\hat{\sigma}$ = 0.85	Type 2	6.78	0.8333	0.1560	0.0322	107.22	53.02

effects the statistical properties accordingly. These networks see the smallest reduction in diameter and increase in clustering. A possible reason for these features may relate to the specific form of the node dynamics and the rate of change of the MSF as α is varied.

Chua based networks displayed the most intricate hubs of all. Each can be seen to be built from nodes connecting to at least two other neighbouring nodes within the same hub. This causes intricate fan-like structures to emerge, increasing the average clustering co-efficient to 0.156, the largest of all our evolved networks. By investing many edges within each hub, these networks also see the lowest density of edges within the centrally connected region.

B. Motif Distributions

With network motifs having been shown to play an important role both functionally [13] and in the classification [14], [15] of many complex systems, we calculated motif distributions for each of our evolved networks. Motifs of size 3 and 4 nodes were considered and detection was performed using the FANMOD algorithm [16]. To assess statistically significant over and under expression of a particular motif, evolved topologies were compared to a sample of 1000 randomised versions, with average expression, z-scores and p-values being calculated. Results are presented in Table II with arrows representing over and under expression of a given motif with a p-value < 0.01.

These distributions highlight two distinct groups of network. The first contains those evolved using the topological eigenratio performance measure. These display little over or under expression of any motif, however, do contain very low frequencies of those with closed loop triangular features. This result can be attributed to minimisation of the eigenratio leading to networks with increased girth – shortest loop in the network. With triangles representing the shortest possible loop, eigenratio evolved networks will remove these features, attempting to maximise their length.

The second group contains all networks evolved using the dynamical order parameter performance measure. By focusing on the statistically significant over and under expression, we see near identical motif distributions, with the only anomaly being the fan-like motif 3. The key feature of these distributions is an over expression of motifs 2, 4 and 6, all closed loop feedback motifs.

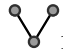
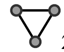
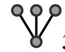
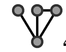
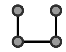
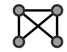
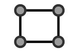
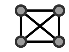
Taking a more detailed look at the actual expression frequencies we also see separation between Type 1 and Type 2 networks. Type 2 networks see a range of frequencies for many of the motifs and in some cases covering frequency seen for dynamically evolved Type 1 networks. This variation is the result of alternative hub-like structures for each type of node dynamic and allows for the specific features that differentiate Type 1 and Type 2 structures to be elucidated. These can be seen in the higher frequencies of motif 3 and 7, and the lower frequencies of motif 5. Together these three motifs appear to be a signature of the Type 2 structure and are therefore likely to be related to hub formation. This seems to confirm the conjecture made in [9] that the presence of feedback motifs is to be expected with higher frequency, independently of the specific node dynamics being considered.

V. CONCLUSIONS

In this work we have outlined a dynamical approach to the evolution of complex networks. Building on previous work in [9], we assessed the generality of Type 2 topologies for networks evolved using the dynamical order parameter

TABLE II

AVERAGE MOTIF FREQUENCIES FOR THE EVOLVED TOPOLOGIES*

Dynamics	Performance Measure	Topology	 1	 2	 3	 4	 5	 6	 7	 8
–	Eigenratio	Type 1	99.90	0.10	19.71 $\downarrow$	0.20	80.05 $\uparrow$	–	0.04 $\downarrow$	–
Rössler	Order Parameter $\hat{\sigma} = 0.6$	Type 1	96.74 $\downarrow$	3.26 $\uparrow$	15.95 $\downarrow$	5.83 $\uparrow$	77.77 $\downarrow$	0.17 $\uparrow$	0.28 $\downarrow$	0.002 $\uparrow$
Rössler	Order Parameter $\hat{\sigma} = 0.9$	Type 2	97.18 $\downarrow$	2.82 $\uparrow$	40.25 $\uparrow$	5.53 $\uparrow$	52.50 $\downarrow$	0.33 $\uparrow$	1.09 $\downarrow$	0.01 $\uparrow$
Lorenz	Order Parameter $\hat{\sigma} = 4.5$	Type 2	98.39 $\downarrow$	1.61 $\uparrow$	25.90 $\downarrow$	3.58 $\uparrow$	69.56 $\downarrow$	0.11 $\uparrow$	0.86 $\downarrow$	–
Chua	Order Parameter $\hat{\sigma} = 0.85$	Type 2	93.98 $\downarrow$	6.02 $\uparrow$	36.74 $\uparrow$	8.60 $\uparrow$	51.93 $\downarrow$	1.36 $\uparrow$	1.28 $\downarrow$	0.10 $\uparrow$

*Arrows represent statistically significant over $\uparrow$ and under $\downarrow$ expression in comparison to a randomised network (p-value < 0.01)

performance measure, at a fixed coupling strength that does not permit full synchronisation, and for a variety of node dynamics. We showed the emergence of Type 2 features for all node dynamics, with the formation hubs of low degree nodes surrounding a highly interconnected central region. Differences in the hub structures were analysed and it was shown that these differences lead to variation in general statistical properties and motif distributions of the evolved networks. Although it is impossible using simulation alone to be sure that such Type 2 features will emerge for every form of node dynamic, analysis of the variation between our different Type 2 topologies allowed for essential shared features to be extracted. From this, two motifs were highlighted having increased and one as decreasing in frequency when compared to Type 1 topologies. These motifs were seen as characteristic of the Type 2 topologies and it was proposed that further analysis of these specific motifs may help understand the role they play in relation to synchronisation. In ongoing work we are also addressing the case of directed network evolution and considering how other factors, such as nonlinear coupling, may also influence the resultant structures.

With dynamics and evolution being a vital component of all complex systems, having tools to understand how various dynamical and topological features arise will become increasingly important. The dynamical approach presented here provides a complementary perspective to those focused solely on topological features; allowing for both dynamics and evolution to be brought together coherently. The question of generality still stands, however, the flexibility of our approach allows for persistent behaviours to be uncovered, providing a promising starting point for more detailed analysis.

VI. ACKNOWLEDGMENTS

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